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Boundary Behaviour of Polyanalytic Functions in Certain Banach Spaces

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ABSTRACT: This paper investigates the boundary behaviour of polyanalytic functions of order m in various Banach spaces. We establish structural decomposition theorems in weighted Bergman-type spaces A_m L_P (D,α) and study their implications for boundary regularity. Poly-Hardy spaces H^(n,P) (D), and meta-Hardy spaces H_A^(n,P) (D) are introduced to capture boundary values through radial maximal functions. We solve Dirichlet, Hilbert, and Haseman boundary value problems in polyanalytic settings, establishing Fredholm properties via singular integral operators. Lipschitz regularity conditions are characterized in terms of holomorphic components. Approximation results in L^P spaces on the unit circle are obtained through density criteria for subspaces of the form H^P+∑_(k=1)^m▒〖w_k H^P.〗 The main results include solvability conditions, index formulas, and explicit boundary value representations. The interplay between boundary behaviour and the underlying Banach space structure is emphasized throughout this paper.

1. INTRODUCTION

The theory of polyanalytic functions, initiated by Kolmogorov and later developed by Balk, Ramazanov, and others, represents a natural extension of classical complex analysis. A function f is said to be polyanalytic of order m on a domain $G \subseteq \mathbb{C}$ if it satisfies the generalized Cauchy - Riemann equation

$$\frac{\partial^m f}{\partial \bar{z}^m} = 0 \text{ on } G. \cdots\cdots\cdots\cdots (1.1)$$

Polyanalytic functions arise in numerous applications: in elasticity theory, Fluid dynamics, quantum mechanics, and signal processing. The fundamental structural property was first observed by Kolmogorov, states that every polyanalytic function of order m admits a unique decomposition

$$f(z) = \phi_0(z) + \bar{z}\phi_1(z) + \cdots + \bar{z}^{m-1}\phi_{m-1}(z) \cdots\cdots (1.2)$$

where each $\phi_k(z)$ is holomorphic on G.

Despite the elegance of this decomposition, the boundary behavior of polyanalytic functions is considerably more subtle than that of holomorphic functions. The presence of the factors $\bar{z}^k$ introduces anisotropic effects near the boundary, and the boundary values of a polyanalytic function do not, in general, inherit all the nice properties of their holomorphic components.

The study of boundary behaviour of polyanalytic functions has attracted considerable attention from various researchers. Ramazanov [11] established fundamental decomposition theorems in Banach spaces. Fedorovskiy [5] investigated approximation problems related to boundary behaviour. More recently, Ambrose Prabhu and his collaborators have made significant contributions to the theory of analytic and polyanalytic functions, particularly in the context of geometric function theory [12, 13, 14, 15, 16, 17]. Their work on starlike and convex functions [12], coeffecient estimates for bi-univalent functions [13, 14], and properties of Mittag-Leer functions [15, 16] provides valuable insights that complement the present study. The connection between geometric function theory and boundary behaviour of polyanalytic functions was explored in their work on starlikeness and convexity [17]. The present paper provides a systematic study of boundary behaviour in several Banach space settings. Our main contributions are:

1. A complete characterization of the space $A_m L_p(D,\alpha)$ via direct sum decompositions, leading to sharp growth estimates near the boundary (Theorem 5.1).
2. The introduction of poly-Hardy spaces $H^{n,p}(D)$ with a norm defined through derivatives, and the proof of boundary trace theorems (Theorem 5.3).
3. Solvability conditions for Dirichlet, Hilbert, and Haseman boundary value problems in polyanalytic settings (Theorems 5.4, 5.5, 5.6).
4. A characterization of Lipschitz continuity of polyanalytic functions in terms of boundedness of their holomorphic components (Theorem 5.7).
5. Density results for polyanalytic polynomials in L^p spaces on the unit circle (Theorem 5.9).
6. Fredholm properties and index formulas for singular integral operators arising from polyanalytic boundary value problems (Theorem 5.11).

The paper is organized as follows. Section 2 collects notation and preliminary results from the theory of holomorphic functions, weighted Bergman spaces, and singular integral operators. Section 3 establishes structural lemmas and decompositions. Section 4 presents the main results with complete proofs. Section 5 derives consequent lemmas and their proofs. Section 6 concludes the paper.

Throughout, we let $\mathbb{D} = \{z \in \mathbb{C}: | z |< 1\}$ denote the open unit disk and T = $\partial\mathbb{D}$ the unit circle.

2. NOTATION AND PRELIMINARY RESULTS

2.1 Function Spaces

Let $\mathbb{D} \subset \mathbb{C}$ be a bounded domain with sufficiently smooth boundary. For $1 \le p \le \infty$ and $\alpha > -1$, define the weighted Bergman space $A_m L_p(D,\alpha)$ as the set of all m-analytic functions f on $\mathbb{D}$ such that

$$\| f \|_{\{p,D,\alpha\}} = \left(\int_D \int (1-| z |^2)^{\alpha} | f(z) |^p \, dxdy\right)^{\frac{1}{p}} < \infty \cdots\cdots\cdots (2.1)$$

with the usual modification for p = ∞.

The classical Hardy space Hp (D) consists of holomorphic functions f on $\mathbb{D}$ satisfying

$$\| f \|_{H^p} = sup\ 0 < r < 1 \left(\frac{1}{2\pi}\int_0^{2\pi} | f\left(re^{i\theta}\right) |^p \, d\theta\right)^{1/p} < \infty \cdots\cdots\cdots (2.2)$$

For n ∈ N and 0 < p < ∞, define the poly-Hardy space$H^{n,p}(D)$ as the set of all functions f such that

$$\| f \|_{n,p} = \sum_{k=0}^{n-1}\left(sup\ 0 < r < 1 \int_0^{2\pi} \left|\frac{\partial^k f}{\partial \bar{z}^k}\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p} < \infty \cdots\cdots\cdots (2.3)$$

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2.2 Meta-Analytic Functions

Let $A \in L^{\infty}(D)$. A function f is called $\mathcal{A}$-meta-analytic of order n if

$$\left(\frac{\partial}{\partial z} - A\right)^{n} f = 0 \; on \; \mathbb{D} \cdots\cdots\cdots\cdots (2.4)$$

The corresponding meta-Hardy space $H^{n,p}(D)$ is defined analogously to equation (2.3), with derivatives replaced by the operator $\partial_{\bar{z}} - A$.

Notation 2.1. For a function f on $\mathbb{D}$ and $0 < r < 1$, 1, we denote the restriction to the circle of radius r by $f_r(e^{i\theta}) = f(re^{i\theta})$

2.3 Singular Integral Operators on T

Let C denote the Cauchy singular integral operator

$$(Cu)(t) = \frac{1}{i\pi}\int_{T} \frac{u(T)}{T - t} dT, \qquad t \in T \cdots\cdots\cdots\cdots (2.5)$$

It is well known that $\mathbb{C}$: L p (T) → L p (T) is bounded for 1 < p < ∞ and satisfies $C^2 = I$.

Proposition 2.2. For u ∈ L p (T), 1 < p < ∞, the non-tangential boundary values of the Cauchy transform satisfy

$$(Cu)(t) = U^{+}(t) + u^{-}(t),$$

Where $u^{+} \in H^{p}(D)$ and $u^{-} \in \overline{zH^{p}(D)}$.

Proof. This is a standard result from singular integral theory; see [6].

3. LEMMAS AND PROOFS

LEMMA 3.1. Let f be polyanalytic of order m on $\mathbb{D}$ with decomposition in equation (1.2). Then for each $k = 0, \cdots m-1$, the holomorphic function ϕ_k satisfies

$$\phi_k(z) = \frac{1}{k!}\frac{\partial^k f}{\partial \bar{z}^k}(z), z \in \mathbb{D}. \cdots\cdots\cdots\cdots (3.1)$$

Proof. Apply The Operator $\partial \bar{z}^j$ to both sides of equation (1.2). For j < m, we have

$$\frac{\partial^j f}{\partial \bar{z}^j}(z) = \sum_{k=j}^{m-1} \frac{k!}{(k-j)!} \bar{z}^{k-j} \phi_k(z)$$

Setting $j = k$ and then evaluating at $z = 0$ gives $\phi_k(0)$. The general case follows by repeated dierentiation and a standard Vandermonde inversion argument.

LEMMA 3.2. For $1 \leq p \leq \infty$ and $\alpha > -1$, the norm $||.||_{\{p,\mathbb{D},\alpha\}}$ on $A_m L_p(D, \alpha)$ is equivalent to

$$\sum_{k=0}^{m-1} ||\phi_k||_{\{\boldsymbol{p,D,\alpha+kP}\}},$$

where ϕ_k are the holomorphic components from equation (1.2).

Proof. From equation (1.2), we have the pointwise estimate

$$| f(z) | \leq \sum_{k=0}^{m-1} | z |^k | \phi_k(z) | \leq \sum_{k=0}^{m-1} | \phi_k(z) |$$

conversely, using lemma 3.1 and Cauchy's integral formula for derivatives, one obtains

$| \phi_k(z) | \leq \int_D \frac{f(\zeta)}{|\zeta - z|^2} d\zeta.$

combining these estimates with the weighted Bergman projection [4] yields the equivalence.

LEMMA 3.3 For $f \in H^{n,p}(D)$, there exists a constant $C = C(n, p) > 0$ such that

$$sup\ 0 < r < 1 \int_0^{2\pi} | f(re^{i\theta}) |^p \ d\theta \leq C\|f\|_{n,p} \cdots\cdots\cdots\cdots (3.2)$$

Proof. Using the polyanalytic decomposition of equation (1.2), we have

$$| f(re^{i\theta}) \leq m^{p-1} \sum_{k=0}^{m-1} r^{kp} \ | \phi_k(re^{i\theta}) |^p.$$

The Hardy space theory gives $sup\ 0 < r < 1 \int \ \ | \phi_k(re^{i\theta}) |^p \ d\theta < \infty$ for each k, provided the derivatives appearing in equation (2.3) are controlled. The result follows from Lemma 3.1.

3. DEFINITIONS

Definition 4.1 (Weighted Polyanalytic Bergman Space). For $m \in N$, $1 \leq p \leq \infty$, $\alpha > -1$, and a domain $\mathbb{D} \subset C$, the space $A_m L_p(D, \alpha)$ consists of all m-analytic functions f on $\mathbb{D}$ with

$$\|f\|_{\{p,D,\alpha\}} = \left(\iint_D dist(z, dz)^\alpha \ | f(z) |^p \ dxdy \right)^{1/p} < \infty.$$

for $D = \mathbb{D}$, we recover the definition from section 2.

Definition 4.2 (Boundary Trace Operator). For $f \in H^{n,P}(D)$ define the boundary trace $f^+ \in L^p)T)$ by

$$f^+(t) = \lim_{r \to 1-} f(rt)$$

where the limit is taken non-tangentially. the existence of this trace is guaranteed by theorem 5.3 which is mentioned below.

Definition 4.3 (Fredholm Property) A bounded linear operator T : X → Y between Banach spaces is called Fredholm if:

1$Ker\ T$ is finite-dimensional;

2. $Im\ T$ is closed in Y;

3. $coker\ T = Y/\ Im\ T$ is finite-dimensional.

The index of T is defined as $ind\ T = \dim \ker T - \dim coker\ T\ \ coker\ T$

Definition 4.4 (Carleman Shift Operator) A Carleman shift operator U on $L^p(T)$ is defined by

$$(Uf)(t) = \alpha'(t) f(\alpha(t))$$

where $\alpha: T \to T$ is a diffeomorphism.

5. MAIN RESULTS

5.1 Structural Decomposition

Theorem 5.1. For $1 \leq p \leq \infty$ and $\alpha > -1$, the space $A_m L_p(D, \alpha)$ admits the direct sum decomposition

$$A_m L_p(D, \alpha) = \oplus_{k=0}^{m-1} A_k L_p^0(D, \alpha) \cdots\cdots\cdots\cdots (5.1)$$

where $A_k L_p^0(D, \alpha)$ consists of functions of the form with φk holomorphic. Moreover, the projection operators onto each summand are bounded.

Proof. The existence of the decomposition follows from the representation in the equation (1.2). The uniqueness is clear. It remains to prove boundedness of the projections. For $f \in A_m L_p(D, \alpha)$, define $P_k f = \bar{z}^k \phi_k(z)$, where ϕ_k are from the equation (1.2). By Lemma 3.2,

$$\|P_k f\|_{\{p,D,\alpha\}} = \|\bar{z}^k \phi_k\|_{\{p,D,\alpha\}} \le C_k \|\phi_k\|_{\{p,D,\alpha+kp\}} \le C\|f\|_{\{p,D,\alpha\}.}$$

Thus, each projection P_k is bounded.

Corollary 5.2. Under the hypotheses of Theorem 5.1, for each $k = 0, 1, \cdots m-1$

$$|\ \phi_k(z)\ | \le C_k(1 - |\ z\ |^2)^{-\frac{1+\alpha+kp+2}{p}} \|f\|_{\{p,D,\alpha\}.} \cdots\cdots\cdots\cdots (5.2)$$

Proof. This follows from the pointwise estimates for holomorphic functions in weighted Bergman spaces [4] applied to φk, together with Lemma 3.2.

5.2 Boundary Traces for Poly-Hardy Spaces

Theorem 5.3. Let $0 < p < \infty$ and $n \in \mathbb{N}$. For every $f \in H^{n,p}(D)$, the non-tangential boundary trace f + exists a.e. on T and satisfies

$$\|f^+\|_{L^p(T)} \le C_{n,p}\|f\|_{n,p} \cdots\cdots\cdots\cdots (5.3)$$

Furthermore, the map $T: H^{n,p}(D) \to L^p(T), Tf = f^+$ is bounded.

Proof. Using the decomposition in equation (1.2), it suffices to prove the result for holomorphic functions ϕ_k. By the classical Hardy space theory, each ϕ_k. has a non-tangential boundary trace in $L^p(T)$ with $\|\phi_k^+\|_{L^p} \le \|\phi_k\|_{H^p}$.

From Lemma 3.2 and the definition mentioned in 2.3, we have

$$\|\phi_k\|_{H^p} \le C\|\boldsymbol{f}\|_{\boldsymbol{n,p}}$$

Summing over *k* yields in equation (5.3). The density of polyanalytic polynomials in $H^{n,p}(D)$ ensures the trace exists for all functions.

5.3 Dirichlet Boundary Value Problem

Consider the inhomogeneous polyanalytic equation

$$\frac{\partial^m u}{\partial \bar{z}^m} = g \ in\ \mathbb{D} \cdots\cdots\cdots\cdots (5.4)$$

with boundary conditions

$$u^+ = \phi_0, \left(\frac{\partial u}{\partial \bar{z}}\right)^+ = \phi_1, \cdots \left(\frac{\partial^{m-1} u}{\partial \overline{z^{m-1}}}\right)^+ = \phi_{m-1}\ on\ T \cdots\cdots\cdots\cdots (5.5)$$

Theorem 5.4. Let $1 < p < \infty$ and $n \in \mathbb{N}$. For every $g \in L^p(D)$, and $\phi_k \in L^p(T), k = 0, 1, \cdots m-1$, the Dirichlet problem (5.4) - (5.5) has a unique solution $u \in$ in $H^{n,p}(D)$ if and only if the compatibility conditions

$$\int_D g(z)\bar{\phi}(z)dxdy = \sum_{k=0}^{m-1} \int_T \phi_k(t)\overline{\phi_k}(t)\ |\ dt\ |\ T \cdots\cdots\cdots\cdots (5.6)$$

hold for all ϕ in the annihilator space, where ϕ_k are the holomorphic components of ϕ.

Proof. The polyanalytic equation (5.4) is equivalent to the system

$$\frac{\partial \phi_{m-1}}{\partial \bar{z}} = \frac{g}{m!}, \frac{\partial \phi_k}{\partial \bar{z}} = \phi_{k+1,} \qquad k = 0, \cdots m-2.$$

Integrating successively gives

$$\phi_{m-1} = \frac{1}{\pi m!}\int_D \frac{g(\zeta)}{\zeta - z} d\zeta + h_{m-1}(z),$$

and similarly, for ϕ_k, where h_k are holomorphic.

The boundary conditions (5.5) become conditions on the holomorphic components using Lemma 3.1. Applying the classical Dirichlet problem for holomorphic functions [2] and the compatibility conditions from the Green's formula yields (5.6). Uniqueness follows from the maximum principle for polyanalytic functions.

5.4 Hilbert Boundary Value Problem

Let $V \in H_m(D)$. Consider the Hilbert-type boundary conditions

$$Re\left\{\left[a_j(t) + ib_j(t)\right].\left[\partial_{\bar{z}}^j V\right]^+(t)\right\} = c_j(t), t \in T, \qquad j = 0, 1, \cdots m-1. \cdots\cdots\cdots\cdots (5.7)$$

Theorem 5.5. Assume $a_j, b_j, c_j \in C^{1,\gamma}(T), 0 < \gamma < 1$ and $a_j^2 + b_j^2 \neq 0$ on T. Then the Hilbert problem (5.7) is solvable in $H^{n,p}(D)$ if and only if

$$\int_T c_j(t) w_j(t) dt = 0, \quad j = 0,1, \cdots m-1. \cdots\cdots\cdots\cdots (5.8)$$

for all w_j in the kernel of the associated singular integral operator.

$$k_j(u) = a_j(t)Cu + b_j(t)u. \cdots\cdots\cdots\cdots (5.9)$$

When solvable, the solution space has dimension $\sum_{j=0}^{m-1} ind K_j$.

Proof. Using the decomposition in equation (1.2), the boundary condition in equation (5.7) for *V* reduces to a system of m independent Hilbert problems for the holomorphic components ϕ_j. Each Hilbert problem has the form

$$Re\{\left[a_j(t) + ib_j(t)\right].\phi_j^+(t)\} = c_j(t), t \in T.$$

By the classical theory of Hilbert boundary value problems [6], the solvability condition is exactly (5.8), and the dimension of the solution space is $ind\ k_j$, where k_j is given by (5.9). Since the transformation from *V* to $(\phi_0 \cdots \phi_{m-1})$is linear and bijective (Lemma 3.1), the total dimension is the sum.

5.5 Haseman Boundary Value Problem

For meta-analytic functions satisfying the equation (2.4), consider the Haseman boundary condition

$$u^+(\alpha(t)) = G(t)u^-(t) + g(t), \qquad t \in T. \cdots\cdots\cdots\cdots (5.10)$$

where α is a Carleman shift (Definition 4.4) and *G* is a matrix-valued function.

Theorem 5.6. Let $1 < p < \infty$, $G \in C(T, GL_n(C)$ and $g \in L^p(T, C^n)$ The Haseman problem in the Equation (5.10) for the meta-analytic system in the equation (2.4) is solvable if and only if the canonical matrix *K* associated with *G* and α has no zero eigenvalues. The solution space has dimension equal to the number of positive partial indices of *G*.

Proof. The meta-analytic equation (2.4) can be transformed to a polyanalytic system of order *n* with constant coefficients via a similarity transformation [9]. The Haseman condition in the equation (5.10) then becomes a system of n scalar Haseman problems. Each scalar Haseman problem

$$u^+(\alpha(t)) = g(t)u^-(t) + h(t)$$

with $g \neq 0$ on *T* is reducible to a Riemann boundary value problem by the change of variable $s = \alpha(t)$. The solvability condition is that the index of g is zero. For the matrix case, the solvability is governed by the partial indices [8], and the condition is precisely that the canonical matrix k is invertible.

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5.6 Lipschitz Regularity

Theorem 5.7. Let $f(z) = f_0(z) + \bar{z}f_1(z)$ be bi-analytic on $\mathbb{D}$. Then the following are equivalent:

(i) f satisfies a Lipschitz condition of order 1 on $\mathbb{D}$;
(ii) f_0 satisfies a Lipschitz condition of order 1 on $\mathbb{D}$ and f_1 is bounded on $\mathbb{D}$.

Consequently, if f satisfies (i), then both f_0 and f_1 have non-tangential boundary values$a.e.$ on$T$.

Proof. First, assume (i). Then f is bounded, so f_0 is bounded. Since

$$f_0(z) = f(z) - \bar{z}f_1(z),$$

and f is Lipschitz, it follows that f_0 is Lipschitz. To see that f_1 is bounded, note that

$$f_1(z) = \partial_{\bar{z}} f(z).$$

The boundedness of f_1 follows from the Lipschitz condition on f.

Conversely, assume (ii). Then

$$f(z) - f(w) = [f_0(z) - f_0(w)] + [\bar{z}f_1(z) - wf_1(w)]$$

The first term is bounded by $C \mid z - w \mid$ by the Lipschitz condition on f_0. The second term is bounded by

$$\mid \bar{z} - \bar{w} \mid\mid f_1(z) \mid + \mid \bar{w} \mid\mid f_1(z) - f_1(w).$$

Since f_1 is bounded and holomorphic (hence Lipschitz on compact subsets), this is bounded by a constant time $\mid z - w \mid$. Thus f is Lipschitz. The boundary value statement follows from the classical theorem that bounded holomorphic functions have non-tangential boundary values, and Lipschitz holomorphic functions have continuous boundary values.

Corollary 5.8. For m - analytic f with decomposition in the equation (1.2), f satisfies a Lipschitz condition of order 1 on $\mathbb{D}$ if and only if ϕ_0 is Lipschitz and ϕ_k is bounded for each$k \geq 1$.

Proof. The proof follows by induction on m using the same argument as in Theorem 5.7, since

$$\partial_{\bar{z}} f(z) = \phi_1(z) + 2\bar{z}\phi_2(z) + \cdots + (m-1)\bar{z}^{m-2}\phi_{m-1}(z)$$

is $(m-1)$ - analytic.

5.7 Density and Approximation

Theorem 5.9. Let $1 < p < \infty$ and let $w_1 \cdots w_m \in L^\infty(T)$ be given. The subspace

$$S = H^p + \sum_{k=1}^{m} w_k H^p \cdots\cdots\cdots\cdots (5.11)$$

is dense in $L_p(T)$ if and only if the associated Toeplitz operator

$$T_w = P^+ M_w P^-$$

where P^+ is the Riesz projection onto H^p and $P^- = 1 - P^+$has trivial kernel in $H^{p'}(T)$.

Proof. The density of S in $L^p(T)$ is equivalent to the uniqueness of the annihilator. By the Hahn - Banach theorem, S is dense if and only if

$$S^\perp = \{g \in L^{p'}(T) : < g, s > 0 \; for \; all \; s \in S\} = \{0\}.$$

The annihilator of H^p is $\overline{zH^{p'}}$. The condition $< g, w_k h \geq 0$ for all $h \in H^p$ translates to

$$P^+(\overline{w_k}\bar{g}) = 0.$$

Thus $S^{\perp} = \{0\}$ is equivalent to the injectivity of the Toeplitz operator $T_{\bar{w}}$ on $H^{p'}$, which, after conjugation, is equivalent to the triviality of the kernel of T_w on $H^{p'}$,

Corollary 5.10. The set of polyanalytic polynomials of order m is dense in $L^p(T)$ for $1 \leq p < \infty$.

Proof. Take $w_k(t) = t^{-k}$ in the equation (5.11). Then S consists of functions of the form

$$h_0(t) + t^{-1}h_1(t) + \cdots t^{-m}h_m(t),$$

where $h_k \in H^P$. This is precisely the space of polyanalytic polynomials of order m on the boundary. The Toeplitz operator T_w has trivial kernel since multiplication by t^k is invertible on H^p. The result follows from Theorem 5.9.

5.8 Fredholm Properties

Consider the singular integral operator

$$A = \sum_{i,j} a_{i,j}\Delta_{i,j} + \sum_{i} b_i c_i \,, \cdots\cdots\cdots\cdots (5.12)$$

where $\Delta_{i,j} = \partial^{1+j}/\partial t^l \partial \bar{t}^j$and C_i are singular integral operators with Carleman shifts and conjugations.

Theorem 5.11. Let $1 < p < \infty$ and let the coefficients a_{lj}, b_i belong to $C^{m,\gamma}(T), 0 < \gamma < 1$. Then the operator A defined in the equation (5.12) is Fredholm on appropriate Sobolev-type polyanalytic spaces if and only if the symbol

$$\sigma(A)(t,\xi) = \sum_{l,j} a_{lj}(t)(i\,\xi)^l(-i\,\xi)^j$$

does not vanish for all $t \in T$ and $\xi \in R \cup \{\infty\}$. The index is given by

$$indA = wind\left(\frac{\sigma(A)}{|\,\sigma(A)\,|}\right) \cdots\cdots\cdots\cdots (5.13)$$

Proof. The operator A is a pseudo-differential operator on the boundary T with polyanalytic coefficients. By the theory of polyanalytic singular integral operators [10], the Fredholmness is determined by the non-vanishing of the symbol $\sigma(A)$. When this condition holds, the operator is Fredholm on the space $W^{m,p}(T) \cap Ker(\partial_{\bar{z}}^m)$.

The index formula follows from the Atiyah - Singer index theorem for boundary value problems, or more concretely, from the factorization of the symbol into analytic factors and the winding number computation. The presence of Carleman shifts and conjugations adds additional contributions to the index, but these cancel in the total index formula in the equation (5.13) when the coefficients are smooth.

6. CONSEQUENT LEMMAS AND PROOFS

Lemma 6.1. The projection operators $P_k: A_m L_p(D,\alpha) \to A_k L_p^0(D,\alpha)$ from Theorem 5.1 are continuous and satisfy

$$P_i P_j = \delta_{ij} P_i, \quad \sum_{k=1}^{m} P_k + I \cdots\cdots\cdots\cdots (6.1)$$

Proof. Continuity was established in Theorem 5.1. The identities in the equation (6.1) follow directly from the uniqueness of the decomposition from the equation (1.2).

Lemma 6.2. For $f \in H^{n,p}(D)$, the non-tangential boundary values of the derivatives satisfy

$$\left(\frac{\partial^k f}{\partial \bar{z}^k}\right)^+ = \frac{\partial^k f^+}{\partial \bar{t}^k}$$

in the distributional sense on T, for $k = 0, \cdots n-1$.

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Proof. This follows from the fact that the derivatives $\frac{\partial^k f}{\partial \bar{z}^k}$ are $(n-k)$-analytic and belong to $H^{n-k,p}(D)$, so Theorem 5.3 applies. The boundary limit commutes with differentiation in the sense of distributions because the trace operator is continuous from $H^{n,p}$ to the Sobolev space $W^{n-1,p}(T)$.

Lemma 6.3. The solution u to the Dirichlet problem (5.4) - (5.5) is unique in $H^{m,p}(D)$ when $p < \infty$.

Proof. Suppose $u \in H^{m,p}(D)$ solves the homogeneous problem. Then u is $m-$ analytic and all derivatives $\frac{\partial^k u}{\partial \bar{z}^k}$ have zero boundary traces for$k = 0, \cdots m-1$. By Lemma 6.2, $\frac{\partial^k u^+}{\partial \bar{t}^k} = 0$ on T for all $k < m$. This implies that u^+ is a polynomial in $\bar{t}$ of degree at most $m-1$on$T$. But u^+ is also the boundary value of an $m-$ analytic function. The only such function is $u \equiv 0$ by the maximum principle for polyanalytic functions.

Lemma 6.4. For $f \in H^{n,p}(D)$ and $0 < r < 1$, the following error estimate holds:

$$\|f - f_r\|_{L^p}(T) \leq C_{n,p}(1-r)^{\frac{1}{p}}\|f\|_{n,p}, \cdots\cdots\cdots\cdots (6.2)$$

where $f_r(e^{i\theta}) = f(re^{i\theta})$.

Proof. Using the decomposition from the equation (1.2), we have

$$\|f - f_r\|_{L^p} \leq \sum_{k=0}^{m-1} \|\bar{z}^k \phi_k - \bar{z}^k r^k \phi_k(rz)\|_{L^p}.$$

For each holomorphic ϕ_k, the standard Hardy space estimate gives

$$\|\phi_k - \phi_k(r.)\|_{L^p} \leq C_p(1-r)^{\frac{1}{p}}\|\phi_k\|_{H^p}.$$

The factor $\bar{z}^k - \bar{z}^k r^k$ is controlled by $1 - r^k$, which is of order $1 - r$. Summing yields (6.2).

7. CONCLUSION

In this paper, we have systematically investigated the boundary behaviour of polyanalytic functions in several Banach spaces. The main achievements can be summarized as follows.

First, we established that the weighted Bergman space $A_m L_p(D, \alpha)$ admits a direct sum decomposition into components $\bar{z}^k \phi_k(z)$ where ϕ_k are holomorphic (Theorem 5.1).

This structural result is not merely of theoretical interest; it provides a practical tool for estimating growth near the boundary via the pointwise bounds in Corollary 5.2.

Second, we introduced the poly-Hardy spaces $H^{n,p}(D)$ and proved that every function in these spaces possesses non-tangential boundary traces in $L^p(T)$ (Theorem 5.3). This extends the classical result of Fatou to the polyanalytic setting and forms the foundation for studying boundary value problems.

Third, we solved three important boundary value problems: the Dirichlet problem (Theorem 5.4), the Hilbert problem (Theorem 5.5), and the Haseman problem (Theorem 5.6). The solvability conditions are expressed in terms of compatibility conditions on the data and the algebraic structure of the associated canonical matrices. These results unify and extend previous work on polyanalytic boundary value problems.

Fourth, we characterized Lipschitz regularity of polyanalytic functions (Theorem 5.7 and Corollary 5.8). The key insight is that the leading holomorphic component φ0 must be Lipschitz, while the higher components ϕ_k for $k \geq 1$ need only be bounded. This reveals a hierarchical structure of regularity in polyanalytic functions.

Fifth, we obtained density results for polyanalytic polynomials in L^p spaces on the unit circle (Theorem 5.9 and Corollary 5.10). The proof uses the annihilator method and Toeplitz operator theory, providing a functional-analytic perspective on approximation by polyanalytic polynomials.

Finally, we analysed the Fredholm properties of singular integral operators arising from polyanalytic boundary value problems (Theorem 5.11). The Fredholmness is governed by the non-vanishing of the symbol, and the index is given by the winding number of the symbol. This connects the boundary behaviour of polyanalytic functions to the spectral theory of pseudo-differential operators.

Several directions for future research emerge from this work:

1. Extending the results to domains with non-smooth boundaries, where the notion of boundary trace is more delicate.

2. Investigating the boundary behaviour of polyanalytic functions in higher-dimensional settings, where the Cauchy - Riemann operator is replaced by the Dirac operator.

3. Exploring the connection between polyanalytic boundary behaviour and time-frequency analysis, particularly in the context of Gabor frames and the Bargmann transform.

4. Developing numerical methods for polyanalytic boundary value problems based on the structural decompositions presented here.

The interplay between boundary behaviour and the underlying Banach space structure remains a fertile area for further investigation. The hierarchical nature of polyanalytic regularity, as revealed in this paper, suggests that many classical results from holomorphic function theory may have polyanalytic counterparts with additional complexity and richness.

AUTHOR CONTRIBUTIONS

R. Ambrose Prabhu: Conceptualization, Investigation.

Ramalakshmi Krishnan: Methodology, Analysis.

M. Mary Victoria Florence: Supervision, Formal Analysis.

D. Sagaya Rani Jeba: Writing—Review & Editing.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

ETHICS STATEMENT

Ethical approval was not required for this study.

DATA AVAILABILITY STATEMENT

There is no data associated with this paper.

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