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Three Coincidence Maps in a Generalized Space of Convex Metric Spaces Determine the Fixed Point and Discuss its Application in the Real World of Medical Sciences

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ABSTRACT: To demonstrate coincidence and fixed points, researchers have employed maximum type generalized non-expansive mappings and conditions for their self-mappings with minimum commutative requirements. We also apply the application in real-world problems. Additionally, we present a novel concept of a fixed function on a metric space. To confirm the efficacy of our findings, some fixed function theorems, illustrative examples, and applications in medical sciences are also provided, "Novel results on a fixed function and their application based on the best approximation of the treatment plan for tumor patients getting intensity modulated radiation therapy (IMRT)" see [15].

1. INTRODUCTION

Gregus Jr. [18] showed in 1980 that non-expansive mappings on a Banach space may lack or have a large number of fixed points, while Banach's contraction principle on a full metric space has a single fixed point. When X occupies a convex space, these maps have a fixed point.

Putting these two ideas together, I came up with a fixed point theorem for non-expansive types of mappings on normed spaces by combining these two notions. With various generalizations and applications. This finding has proven to be highly useful with various generalizations and applications (see, under references [1], [5]-[13], [17], [26], [27] [33-36] and [39-41].

The coincidences and fixed points of three maps on an arbitrary set are the main subjects of this research. We're interested in the fact that they satisfy the non-expansive generalized types criterion. Both Singh and Mishra [31] are expansions and generalizations of Singh and Mishra [30], which is based on C'iric, Delbosco et al., Fisher, Gregus Jr., and Li [9, 17, 18, and 26].

We will need Das and Naik to show us how to improve the fixed point theorem (see remark (b) below).

Definition1.1. [see 16]**:** Assume (U, d) is a metric space, P is a maps of X to itself and q is a non-negative real number such that the inequality $d(Pm, Pn) \leq q\, d(m, n)\ \ \forall m, n \in U$. If $q < 1$, then P is a contraction mapping; if $q = 1$ then P is a non-expansive mapping.

Theorem A. (Singh [29]). Assume V be an arbitrary non-empty set and (U, d) a metric space. Assume $P, Q : V \rightarrow U$ be such that $P(Y) \subset Q(Y)$ and the following is satisfied for some $0 < q < 1$ and for all $m, n \in V$.

$$d(Pm, Pn) \leq q \max\{(d(Qm, Qn),\ d(Qm, Pn), d(Qn, Pn), d(Qm, Pm) d(Qn, Pm) \quad (1)$$

If $P(Y)$ or $Q(Y)$ is a complete subspace of U then there exists an elements $z \in V$ such that $Qz = Pz$, i.e. z is a coincidence point of P and Q further,

(I) If there exists $v, w \in V$ such that $Qv = Pv$ and $Qw = Pw$, then

$Qv = Qw$;

(II) If $V = U$ and P, Q commute at their coincidence point z then P and Q have a unique common fixed point.

Theorem A is derived from Corollary 2 of Singh and Mishra [30] as well.

Remarks: (a) C'iric' quasi-contraction [4], the most general among contractive type conditions (cf. Das and Naik [14] (p. 369)) is obtained from (1) when $V = U$ and Q is the identity map on U.

(b) The main result (Theorem 2.1) of Das and Naik [14] is obtained under the condition (1) with $V = U$ and P continuous.

Definition 1.2 (Takahashi [37])**:** Assume (U, d) be a metric space and $I = [0,1]$. A continuous mapping $W: U \times U \times I \rightarrow U$ is a convex structure on U if the inequality.

$d(u, W(m,n,\lambda)) \leq \lambda\, d(u,m) + (1-\lambda)\, d(u,n),\ \forall m,n,u \in U\ and\ \lambda \in I$. The metric space U together with a convex structure is called a convex metric space. A subset K of U is convex if $W\ (m,n,\lambda) \in K$ for all m, n in K and λ in I.

If U is compact, then the continuity assumption on W becomes superfluous (see Talman [38]). Krik [20] used the phrase "hyperbolic type space" see also Ding [16], Singh et al [32].

2. FIXED POINT THEOREM

Our main aim is to find the following theorem for three single-valued mappings that was obtained by S. L. Singh and S. N. Mishra [30] for two single-valued mappings.

Theorem 2.1: Assume V be an arbitrary non-empty set and (U, d) a convex metric space. Assume Q, S, P be mappings from V to U such that $S(V) \cup P(V) \subseteq Q(V)$ and $Q(V), S(V), P(V)$ are a complete subspace of U. If a, b, c is non-negative real number such that.

$$d(Sm,Pn) \leq a\, d(Qm,Qn) + b \max\{d(Qm,Sm), d(Qn,Pn)\} + c[d(Qm,Pn) + d(Qn,Sm)] \qquad (2)$$

$$for\ all\ m,n\ \in V\ where\ 0 < a,b,c\ < 1 \qquad (3)$$

$$and\ a + b + 2c\ \leq 1 \qquad (4)$$

then Q, S and P have a coincidence point $u \in V\ such\ that\ Qu = Su = Pu$ further,

(i) if there exist $v, w \in V\ such\ that\ Qv = Sv = Pv$ and $Qw = Sw = Pw$ then $Qv = Qw$.

(ii) If V is a subset of U and Q, S or P commute at their coincidence point u then Q, S and P have a unique common fixed point.

Moreover, if (4) hold with the strict inequality, then the convexity, assumptions on U, as well as the condition (3) can be omitted with the proof is prefaced by the following Lemma

Lemma 2.1: Assume V be an arbitrary non-empty set and mappings from V to U, such that,

$S(V) \cup P(V) \subseteq Q(V)$, If a, b, c are non-negative real numbers such that (2), (3) and

$$\mathrm{a} + \mathrm{b} + 2\mathrm{c} = 1 \qquad (5)$$

Hold then,

(i) $inf\ \{d(Qm,Sm), d(Qm,Pm) : m \in V\} = 0$ and

(ii) $max\{d(Qm,Qn), d(Sm,Pn)\} \leq (1 + a + 2b).M/b$

where $M = max\ \{d(Qm,Sm), d(Qm,Pm)\}$

Proof: Consider a point $m_0 \in V$. Then there exists an element $m_1 \in V$ such that

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$fm_1 = Sm_0$. Such a choice is permissible since $S(V) \cup P(V) \subseteq Q(V)$. Similarly, choose $m_2, m_3, m_4 \in V$. Such that $Qm_2 = Pm_1, Qm_3 = Sm_2, Qm_4 = Pm_3, Qm_5 = Sm_4$ and $Qm_6 = Pm_5, Qm_{2i+1} = Sm_{2i}, Qm_{2i+1} = Pm_{2i+1}$.

Set $r_i = d(Qm_i, Qm_{i+1}),\ O_i = d(Sm_{2i}, Pm_{2i+1}), i = 0, 1, 2, 3, \ldots$

Then from (2)

$r_2 = d(Qm_2, Qm_3) = d(Sm_2, Pm_1)$

$\leq a\, d(Qm_2, Qm_1) + b \max\{ d(Qm_2, Sm_2), d(Qm_1, Pm_1)\} + c\,[d(Qm_2, Pm_1) + d(Qm_1, Sm_2)]$

$= a\, d(Qm_2, Qm_1) + b \max\{ d(Qm_2, Qm_3), d(Qm_1, Qm_2)\} + c\,[0 + d(Qm_1, Qm_3)]$

$\leq a\, d(Qm_2, Qm_1) + b \max\{ d(Qm_2, Qm_3), d(Qm_1, Qm_2)\} + c\,[d(Qm_1, Qm_2) + d(Qm_2, Qm_3)]$

$$r_2 \leq a\, r_1 + b \max\{r_1, r_2\} + c\,[r_1 + r_2] \quad (6)$$

If $r_{i-1} < r_i$, for some i,

we have $r_1 < r_2$,

then (6) implies $r_1 < (a + b + 2c)\, r_2 = r_2$,

is a contradiction, therefore $r_2 \leq r_1$ and similarly $r_1 \leq r_0$.

Thus $r_2 \leq r_0 = d(Qm_0, Qm_1)$,

from (2) again,

$O_2 = d(Sm_4, Pm_5)$

$\leq a\, d(Qm_4, Qm_5) + b \max\{ d(Qm_4, Sm_4), d(Qm_5, Pm_5)\} + c\,[d(Qm_4, Pm_5) + d(Qm_5, Sm_4)]$

$= a\, d(Qm_4, Qm_5) + b \max\{ d(Qm_4, Sm_4), d(Qm_5, Pm_5)\} + c\,[d(Qm_4, Pm_5) + 0]$

$O_2 \leq a\, r_4 + b \max\{r_4, r_5\} + c\,[r_4 + r_5]$,

$O_2 < (a + b + 2c) r_0 = r_0$.

Similarly $O_1 \leq r_0$ and

$$\max\{O_1, O_2\} \leq r_0 \quad (7)$$

Therefore from (6), $r_2 \leq r_0$.

If $c > 0$, then

$d(Qm_2, Qm_3) \leq d(Qm_0, Qm_1)$.

If $c = 0$, then (5) becomes $a + b = 1$.

Consider

$$Qz = W(Sm_2, Sm_4, 1/2) \quad (8)$$

since V is convex, $z \in V$

$$d(Qz, Pz) \leq {}^{1}/_{2}\ d(Sm_2, Pz) + {}^{1}/_{2}\ d(Sm_4, Pz) \quad (9)$$

$$d(Sm_2, Pz) \leq a\, d(Qm_2, Qz) + b \max\{ d(Qm_2, Sm_2), d(Qz, Pz)\} \quad (10)$$

and

$$d(Sm_4, Pz) \leq a\, d(Qm_4, Qz) + b \max\{ d(Qm_4, Sm_4), d(Qz, Pz)\} \quad (11)$$

therefore, from (9), (10) and (11),

$$d(Qz, Pz) \le {}^1/_2\,[a\, d(Qm_2, Qz) + b\, max\{d(Qm_2, Sm_2), d(Qz, Pz)\}]$$
$$+ {}^1/_2\,[a\, d(Qm_4, Qz) + b\, max\{d(Qm_4, Sm_4), d(Qz, Pz)\}]$$
$$\le {}^1/_2\,[a\, d(Qm_2, Qm_4) + b\, max\{d(Qm_2, Qm_3), d(Qz, Pz)\}$$
$$+ b\, max\{d(Qm_4, Qm_5), d(Qz, Pz)\}]$$
$$\le {}^1/_2\,[a\,\{d(Qm_2, Qm_3) + d(Qm_3, Qm_4)\} + b\, max\{r_2, d(Qz, Pz)\}$$
$$+ b\, max\{r_4, d(Qz, Pz)\}]$$
$$\le {}^1/_2\,[a\,(r_2 + r_3) + b\,(r_2 + r_4)] \le (a + b)\, r_0.$$

Therefore, $d(Qz, Pz) \le r_0$.

Therefore from (8) we conclude that either $c > 0$ or $c = 0$, there exists a points $n \in V$, such that,

$$\mathrm{d(Qn, Pn)} \le \mathrm{d}(\mathrm{Q}m_0, \mathrm{Q}m_1) \qquad (12)$$

Similarly

$$\mathrm{d(Qn, Sn)} \le \mathrm{d}(\mathrm{Q}m_0, \mathrm{Q}m_1) \qquad (13)$$

Both (12) and (13) imply

$inf\,\{d(Qm, Sm), d(Qm, Pm)\} \le d(Qm_0, Qm_1)$.

This proved (III). To prove (IV), we have

$$d(Qm, Qn) \le d(Qm, Sm) + d(Sm, Pn) + d(Qn, Pn)$$
$$d(Qm, Qn) \le d(Sm, Pn) + 2M \qquad (14)$$

now,

$$d(Qm, Pn) \le d(Qm, Sm) + d(Sm, Pn)$$
$$d(Qm, Pn) \le d(Sm, Pn) + M \qquad (15)$$

and,

$$d(Qn, Sm) \le d(Qn, Pn) + d(Sm, Pn)$$
$$d(Qn, Sm) \le d(Sm, Pn) + M \qquad (16)$$

Using (14), (15) and (16), we have

$$d(Sm, Pn) \le a\, d(Qm, Qn) + b\, max\,\{d(Qm, Sm), d(Qn, Pn)\} + c\,[d(Qm, Pn) + d(Qn, Sm)]$$
$$\le a\,[d(Sm, Pn) + 2M] + b\,M + c\,[M + d(Sm, Pn) + M + d(Sm, Pn)]$$
$$\le a\,[d(Sm, Pn) + 2M] + b\,M + 2c\,[M + d(Sm, Pn)]$$
$$\le (2a + b + 2c).M + (a + 2c)\, d(Sm, Pn)$$
$$\le (1 + a)\, M + (1 - b)\, d(Sm, Pn)$$
$$d(Sm, Pn) \le (1 + a)M/b \qquad (17)$$

then from (14) and (17), we get

$$max\{d(Qm, Qn), d(Sm, Pn))\} \le (1 + a + 2b).M/b$$

where $M = max\,\{d(Qm, Sm), d(Qm, Pm)\}$, this completes the proofs.

Proof of Theorem 2.1: If the strict inequality in (4) hold, then (2) implies (1) therefore in this case Theorem 2.1 follows from Theorem (A) condition (2) and convexity assumptions are not needed, so we proceed with the equality condition in (4).

In view of Lemma 2.1 (i),

$$inf\ \{d(Qm, Sm), d(Qm, Pm) : m \in V\} = 0.$$

We choose a sequence $m_j \in V$ such that

$$d(Qm_i, Pm_i) \leq {}^{1}/_{j}\ , \qquad j = 1, 2, \dots \dots .\qquad (18)$$

for $1 \leq j \leq k, k \geq 1 \in K$ is real number, Lemma 2.1 (ii) implies that

$max\{d(Qm, Qn), d(Sm, Pn)\} \leq \{(1 + a + 2b). M/b\}. (1/n).$

Therefore, sequence $\{Qm_j\}$, $\{Sm_j\}$ and $\{Pm_j\}$ are Cauchy sequence in $Q(V)$. Suppose that $Q(V)$ is complete, and then in view of (18), there is a point $h \in Q(V)$ such that sequence $\{Qm_j\}$, $\{Sm_j\}$ and $\{Pm_j\}$ converges to h . Then there exists a point $u \in V$ such that $u \in Q^{-1}h$, thus

$h = Qu$. To prove $h = Pu$, taking $m = m_j$ and $n = u$ in (2) and passing over to the limit, we get,

$d(Sm_j, Pu) \leq a\, d(Qm_j, Qu) + b\, max\, \{d(Qm_j, Sm_j), d(Qu, Pu)\} + c\, [d(Qm_j, Pu) + d(Qu, Sm_j)]$

$d(h, Pu) \leq b\, d(h, Pu) + c\, d(h, Pu) = (b + c)d(h, Pu)$

$= (1 - (b + c))\, d(h, Pu) \leq 0$.

This yields $d(h, Pu) = 0 \Rightarrow h = Pu$, since $0 < b + c < 1$,

Similarly, if we suppose

$inf\ \{d(Qm, Sm) : m \in U\} = 0$, then we obtained $h = Su$.

Further, if we assume the completeness of $Q(V)$ then we get the same conclusion since

$S(V) \cup P(V) \subseteq Q(V)$. To prove (I), take $m = v$ and $n = w$ in (2) and simplify

$$d(Sv, Pw) \leq a\, d(Qv, Qw) + b\, max\, \{d(Qv, Sv), d(Qw, Pw)\} + c\, [d(Qv, Pw) + d(Qw, Sv)]$$

$d(Qv, Qw)\, (1 - (a + 2c)) \leq 0.$

$b\, d(Qv, Qw) \leq 0.$

This in view of (3) gives $Qv = Qw$.

Similarly, we can prove that $Sv = Sw$ and $Pv = Pw$.

To prove (II), notice that the commutativity of P and Q at u and $Qu = Pu$, imply,

$$QQu = Q(Pu) = P(Qu) = PPu$$

now commutativity of S and Q at u and $Qu = Su$ imply,

$QQu = Q(Su) = S(Qu) = SSu.$

Then from (2),

$d(QQu, Qu) \leq a\, d(Qu, Qu) + b\, max\, \{d(Qu, QQu), d(Qu, Qu)\} + c\, [d(Qu, QQu) + d(Qu, QQu)]$

$\leq b + 2c$

$d(QQu, Qu) \leq d(SSu, Pu) \leq a\, d(QQu, Qu) + b\, max\, \{d(QQu, SSu), d(Qu, Pu)\} + c\, [d(QQu, Pu) + d(fu, SSu)]$

$\leq (a + 2c)\, d(QQu, Qu) = (1 - b)\, d(QQu, Qu).$

Yielding $QQu = Qu$. Thus, Qu is a common fixed point of S and P.

Similarly, we have obtained Su and Pu is a common fixed point of respectively Q, P and Q, S. The uniqueness of the point it follows easily.

Corollary 2.1: Assume V be an arbitrary non-empty set and (U, d) a convex metric space. Assume $Q, S, P: U \rightarrow V$, such that $S(V) \cup P(V) \subseteq Q(V)$ and $Q(V), S(V), P(V)$ are a complete subspace of U. If a, b, c are non-negative real numbers such that (3), and

$$a + 2b + 2c \leq 1 \qquad (19)$$

$$d(Sm, Pn) \leq a\, d(Qm, Qn) + b\,\{d(Qm, Sm) + d(Qn, Pn)\} + c[d(Qm, Pn) + d(Qn, Sm)] \qquad (20)$$

Hold for all $m, n \in V$, then Q, S, P have a unique common fixed point in Y.

Proof: Proof is follows as Theorem 2.1 with used condition (19) besides condition (4).

Corollary 2.2: Assume V be an arbitrary non-empty set and (U, d) a convex metric space. Assume $Q, S, P: V \rightarrow U$, such that $S(V) \cup P(V) \subseteq Q(V)$ and $Q(V), S(V), P(V)$ are a complete subspace of U. If a, b, c are non-negative real numbers such that (3), (4) and

$$d(Sm, Pn) \leq a\, d(m, n) + b\,\{d(m, Sm), d(n, Pn)\} + c[d(m, Pn) + d(n, Sm)] \qquad (21)$$

Hold for all $m, n \in V$, then Q, S, P have a unique common fixed point in V.

Proof: Proof of Corollary 2.2 is same as Theorem 2.1, if we are take $Q = I$ the identity mapping then equation (2) is becomes as Corollary 2.2

Corollary 2.3: Assume V be an arbitrary non-empty set and (U, d) a convex metric space. Assume $P: V \rightarrow U$, such that $P(V) \subseteq V$ and $P(V)$ are a complete subspace of U. If a, b, c are non-negative real numbers such that (3), (4) and

$$d(Pm, Pn) \leq a\, d(m, n) + b\, Max\{d(m, Pm), d(n, Pn)\} + c[d(m, Pn) + d(n, Pm)] \qquad (22)$$

Hold for all $m, n \in V$, then P has a unique common fixed point in P.

Proof: If we are take $Sm = Pm$ and $Q = I$ the identity mappings in (2) then Corollary 2.3 becomes like a Corollary of C'iric' [7].

Corollary 2.4: Assume C be a complete convex metric space and $Q: U \rightarrow U$, such that $Q(U) = U$ and $S(U) = P(U) =$ Identity mappings, and

$$d(m, n) \leq a\, d(Qm, Qn) + b\, Max\{d(m, Qm), d(n, Qn)\} + c[d(n, Qm) + d(m, Qn)] \qquad (23)$$

For all $m, n \in U$, where the non-negative numbers a, b, c satisfy (3) and (4). Then Q has a unique fixed point.

Example 2.1: Assume $U = [0, 1]$ with the usual metric d and

$$Qm = \begin{Bmatrix} 3/4, if\ m \in \left[0, \frac{1}{2}\right) \\ 1,\ if\ m \in [\frac{1}{2}, 1] \end{Bmatrix}$$

$$Sm = \begin{Bmatrix} 0,\ if\ m \in \left[0, \frac{1}{2}\right) \\ 1,\ if\ m \in [\frac{1}{2}, 1] \end{Bmatrix}$$

$$Pm = \begin{cases} 0, & if\ m \in \left[0, \frac{1}{2}\right) \\ 1/2, & if\ m \in [\frac{1}{2}, 1] \end{cases}$$

Taking $m = 0, n = 1/2$ in (1) with $Q = I$ the identity map on $V = U$ implies $q \geq 1$, a contradiction to $q < 1$. The mapping P does not satisfy the hypotheses of Corollary 2.3 as well. Indeed, for $m = 1/4$ and $n = 1/2$. Corollary 2.2 yields

$1/2 \leq (a + b + 3c)/4 \leq (1 + c)/4$, since $a + b + 2c \leq 1$.

This contradicts $0 \leq c \leq 1/2$.

Now if $m \in [0, 1/2)$ and $n \in [0, 1/2)$ or $n \in [0, 1/2)$ and $m \in [0, 1/2)$, then considering

$a + b + 2c = 1$, we have

$= a\, d(Qx, Qy) + b\, [d(Qx, Px) + d(Qy, Py)] + c\, [d(Qx, Py) + d(Qy, Px)]$

$= a\, d(1, 3/4) + b\, max\, \{d(3/4, 0), d(1, 1/2)\} + c\, [d(3/4, 1/2) + d(1, 0)]$

$= a/4 + 3b/4 + 5c/4$

$= (a + 3b + 5c)/4 > 1/2 = d(Sx, Py)$.

Thus, (2) -(4) are satisfied.

Example 2.2: Assume $U = (-\infty, \infty)$ with the usual metric d and $Qm = m + 2, Sm = m, Pm, = m, m \in U$. Then

$$d(Qm, Pn) + d(Qn, Sm) = |2 + m - n| + |2 + n - m|$$

$$= \begin{cases} 2d(m, n), if\ d(m, n) \geq 2 \\ 4, \ otherwise \end{cases}$$

Therefore, with $b = 0$ and $a + 2c = 1$,

$$a\, d(Qm, Qn) + c\, [d(Qm, Pn) + d(Qn, Sm)] \geq (a + 2c)\, d(m, n) = d(Sm, Pn).$$

Thus Q, S and Q satisfy (2) with equality in (4) and $b = 0$.

Nevertheless, commuting and continuous mappings Q, S and P do not have a coincidence even though every point of U is fixed under S and P. Notice that S and P does not satisfy (2) along with (3) and (4) as well.

Example 2.3: Assume U be the set of real numbers and $K = U$. Define the mappings S and P to itself by $Sm = (m + 1)/6 = Tm$, then we have

$d(Sm, Pn) = 1/6\, .\, d(m, n)$ and

$\{d(m, Qn) + d(n, Sm)\} = \begin{cases} 2d(m, n), if\ d(m, n) \geq 1 \\ 2, \ otherwise \end{cases}$

Therefore, mappings S and P satisfies condition of Corollary 2.2 with inequality in (4) and $a = 1/6, b = 0, c = 1/6$ and $a + 2c = 1, m = 0, n = 1$. Nevertheless, S, P does not have a fixed point.

Example 2.4: Assume $U = [0, \infty)$ with the usual metric d and

$Qm = 2m, \ m \in U$, then Q does not satisfy Corollary 2.4. Indeed for $m = 0, n = 1$, Corollary 2.4 implies $1 \leq a + b + 3c$, a contradiction to $c \geq 0$. However,

$$a\, d(Qm, Qn) + b\, max\, \{d(m, Qm), d(n, Qn)\} + c\, [d(m, Qn) + d(n, Qm)]$$

$$\geq 2a\, d(m, n) + b\, max\, \{m, n\} + c\, [d(n, 2m) + d(m, 2n)]$$

$$= 2a\, d(m,n) + b\, d(m,n) + 2c\, d(m,m)$$

$$\geq (2a + b + 2c)\, d(m,n).$$

Evidently, Corollary 2.4 ensures a unique fixed point at $m = 0$. Notice also that P is not a quasi-contraction since by taking $m = 0, n = 3$ and $Pm = 2m$ in equation (1) with $V = U$ and $Q =$ the identity mappings, one gets the contradiction $q \geq 1$

3. APPLICATION

Many of the problems that researchers are trying to solve in the real world are nonlinear. Scholars have started to adapt the structure of convexity to non-linear spaces. For example, Takahashi [37], Kirk [20, 21, 22, 23], Meena Joshi et al. [19] and Penot [28] developed this concept in metric spaces.

A metric space (U, d) together with a continuous mapping $W: U \times U \times I \to U$ is a convex structure on U if the inequality.

$d(u, W(m,n,\lambda)) \leq \lambda\, d(u,m) + (1-\lambda)\, d(u,n)$ holds for all m, n, u in U and $\lambda \in I = [0,1]$.

A subset V of U is called convex if $W(m,n,t) \in V$, for all $m, n \in V$ and all $t \in I$.

Example 3.1. Assume $U = M_2(R)$. Matrix A and B are being shown as if they were for any value.

$$A = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \; and\; B = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \; and\; t \in I = [0,1],$$

We define the mapping $W : U \times U \times I \to U$

by $W(A,B,t) = \begin{bmatrix} tc_1 + (1-t)d_1 & tc_2 + (1-t)d_2 \\ tc_3 + (1-t)d_3 & tc_4 + (1-t)d_4 \end{bmatrix}$

and mapping $d : U \times U \to [0, +\infty)$ by $d(A,B) = \sum_{i=0}^{4} |c_i - d_i|$ so , (U, W, d) is a convex metric space. This approach can be used to solve a matrix problem.

Example 3.2. Let $U = R^2$ with the metric

$$d((m_1, m_2), (n_1, n_2)) = max\, \{|m_1 - n_1|, |m_2 - n_2|\}$$

for every $(m_1, m_2), (n_1, n_2) \in U$ and the mapping $W : U \times U \times I \to U$

$$W((m_1, m_2), (n_1, n_2), t) = tm_1 + (1-t)n_1,\ tm_2 + (1-t)n_2$$

For every $(m_1, m_2), (n_1, n_2) \in U$ and $t \in I = [0,1]$, (U, W, d) is a convex metric space, we use this concept in geometry problem. (see 23)

Example 3.3. Let $U = C([0,1])$ be the metric space with the metric

$$d(P,Q) = \int_0^1 |P(m) - Q(m)|\, dm \qquad (24)$$

and mapping $W : U \times U \times I \to U$

with convexity structure $W(P,Q,t) = tP + (1-t)Q$, for every $P, Q \in U$ and $t \in [0,1]$.

Then (U, W, d) is a convex metric space,

By definition 1.1, $d(u, W(m,n,\lambda)) \leq \lambda\, d(u,m) + (1-\lambda)\, d(u,n)$ holds for all m, n, u in U and λ in $I = [0,1]$.

So we can write

$d(Sm, W(Pm, Qm, \lambda)) \leq \lambda\, d(Sm, Pm) + (1-\lambda)\, d(Sm, Qm)$

holds for all m, n, u in U and λ in $I = [0,1]$. Satisfied the convex structure with metric space (U, d).

let $Pm = 2m, Qm = 2m + 1, Sm = m^2$ so we can get with help of (24)

$d(Sm, Pm) = 2/3$ and $d(Sm, Pm) = 5/6$, we calculate

$$d(Sm, W(Pm, Qm, \lambda)) \leq \lambda\, d(Sm, Pm) + (1-\lambda)\, d(Sm, Qm)$$

$$\leq 2\lambda/3 + 5(1-\lambda)/6 = (5-\lambda)/6.$$

This concept we can use in integral calculation problem.

Convex structure is a derivation of convex hull in collinear spaces that also provides good results that appear to be only achievable in linear spaces. Iterative approximation, all types sequences of fixed points in metric spaces developed useful applications by many researchers which need the space to be linear or convex. Refer to [2, 3, 19-23, 28] for more information.

4. APPLICATION IN MEDICAL SCIENCES

For tumor patients receiving intensity modulated radiation therapy (IMRT), the application in this part is based on the best approximation of the treatment plan. This method greatly increases the accuracy of the results by using a proper dose deposition coefficient (DDC) matrix truncation. To estimate the dose distribution to each voxel in the necessary volume of interest from each beamlet with unit intensity, a DDC matrix is frequently calculated. However, during calculations, we typically receive a significant set of data that necessitates a large amount of computer memory and computing efficiency. Consequently, the quality of the treatment plan is impacted since small values from the DDC matrix are typically truncated.

An extremely effective and efficient method for resolving this issue is the fixed point iteration method. In this technique, a proper DDC matrix truncation is used, which significantly improves the accuracy of results see [15].

5. CONCLUSION

We see in proof of Corollary 2.1 is follows as Theorem 2.1 with used condition (19) besides condition (4) then Q, S, P have a unique common fixed point in Y. We see in proof of Corollary 2.2 is same as Theorem 2.1, if we are take $Q =$ the identity mapping then equation (2) is becomes as Corollary 2.2. If we are assume $Sm = Pm$ and $Q =$ The identity mappings in equation (2) of Theorem 2.1 then Corollary 2.3 becomes like a Corollary of C'iric' [7]. If we assume $Q(U) = U$ and $S(U) = P(U) =$ Identity mappings, and for all $m, n \in U$, where the non-negative numbers a, b, c satisfy equation (3) and equation (4). Then Q has a unique fixed point it becomes Corollary 2.4. and we discussed application in real world problem. Nevertheless, there is currently no definition for the term "fixed function." This new expanded idea can be used in a variety of ways in the future to demonstrate the existence and uniqueness of fixed functions, such as by altering the nature of mappings, utilizing different contractive conditions, altering the space, or utilizing different topological structures. The application that enables us to diagnose multiple patients simultaneously directly contributes to this outcome's efficacy.

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