Natural Resources for Human Health
eISSN: 2583-1194

Original Research

Received 04 June 2026
Revised 24 July 2026
Accepted 05 August 2026

Natural Resources for Human Health 2026, Vol. 6 (6): 493–498

KEYWORDS:
groundwater; arid regions; contaminant transport; Mean Breakthrough Time; Caputo-Fabrizio derivative; advection-dispersion equation.

Caputo-Fabrizio Fractional Advection-Dispersion Model for Contaminant Transport in Groundwater

Bouzenna Fatma El Ghenbazia[1], Meftah Mohamed Tayeb[2]

[1]Department of Physics and UDERZA Research Unit, Faculty of Exact Sciences, University of Chahid Hamma Lakhdar, 39000, El Oued, Algeria
[2]Department of Physics and LRPPS laboratory, Faculty of Mathematics and Matter Sciences, University KasdiMerbah, Ouargla 30000, Algeria

ABSTRACT: Groundwater is often the only reliable water supply in arid regions, so predicting how a dissolved contaminant spreads in a shallow aquifer matters for protecting it. The classical advection-dispersion equation (ADE) has a memoryless time derivative and does not reproduce the early arrival and long tails often seen in field breakthrough curves. Here the time derivative of one -dimensional ADE is replaced by the Caputo-Fabrizio (CF) fractional derivative, whose exponential kernel is non-singular and gives a fading memory controlled by the order α. For Semi-infinite column with a constant-concentration inlet, the problem is solved exactly in the Laplace domain and inverted numerically with the Talbot method. For α=1 the result matches the classical Ogata-Banks solution to machine precision. We show analytically that the mean breakthrough time is exactly 1/ α times the advective time, so α acts as a retardation factor, and that the model has a non-zero early-time response, given in closed form, and a heavier tail than the classical case. At a Péclet number of 10, lowering α from 1 to 0.5 delays the half-concentration arrival time by about 48%, from 0.91 to 1.35 advective time units. The model is analytically tractable, but it still has to be calibrated on tracer-test data before it can be used for risk screening.

1. INTRODUCTION

Lorem Groundwater is the main source of drinking and irrigation water in most Saharan regions, and shallow unconfined aquifers are expose to contamination from untreated wastewater, agricultural return flow, and, in some areas, industrial discharge. Predicting solute transport is therefore a basic need in water-resource management and risk assessment(Bear, 1972). The advection-dispersion equation (ADE) is the standard model, and its solution for a semi-infinite column with a constant-concentration inlet was given by Ogata and Banks (1961). The ADE assumes that the time derivative is local and memoryless, and that dispersion follows Fick's law with a constant dispersion coefficient. These assumptions are reasonable for homogeneous laboratory columns, but they often fail in field-scale aquifers, where heterogeneity, preferential pathways, and mass-transfer limitations produce non-Fickian behavior. Tracer experiments in natural porous media frequently show early arrival of the and long tails in the breakthrough curve, which the classical ADE cannot reproduce (Benson et al., 2000; De Moraes et al., 2022; Schumer et al., 2003). Such anomalous transport has been modeled in several ways. The mobile-immobile (MIM) model divides the pore space into mobile and immobile regions and introduces a first-order mass transfer between them; it captures tailing but requires additional parameters that are difficult to identify. Continuous-time random walks (CTRW) provide a stochastic framework in which the waiting-time distribution controls the shape of the breakthrough curve; the classical ADE corresponds to a waiting time with finite mean, while heavy-tailed waiting times produce anomalous transport. Fractional-order equations, in which a derivative is replaced by a non-integer one, offer a deterministic alternative. The fractional advection-dispersion equation (FADE) with a Caputo or Rieman-Liouville time derivative produces subdiffusion, while a fractional spatial derivative produces superdiffusion and skewness (Benson et al., 2000; Schumer et al.,

2003). The choice of fractional operator is not unique, and different operators emphasize different aspects of memory and non-locality. The Riemann-Liouville derivative is defined through a fractional integral and is convenient in pure mathematics, but its initial conditions are fractional integrals of the function and are difficult to interpret physically. The Caputo derivative uses a power-law kernel and allows classical integer-order initial conditions, which is why it has been widely used in transport modeling (Benson et al., 2000; Schumer et al., 2003). However, both Riemann-Liouville and Caputo derivatives have a singular kernel at the origin, which can complicate numerical schemes and physical interpretation. Other non-singular kernels such as the Atangana-Baleanu operator(Atangana and Baleanu, 2016), have also been proposed and applied to transport problems.

The Caputo-Fractional (CF) derivative, introduced in 2015 (Caputo and Fabrizio, 2015; Losada and Nieto, 2015), replaces the singular power-law kernel with an exponential one. The memory is non-singular and fades exponentially over a time set by the order α, and the Laplace transform of the operator is a simple rational function. This last property is particularly attractive for analytical work: it allows the governing equation to be transformed into an ordinary differential equation in the Laplace domain, which can often be solved exactly. The derivative has been used for diffusion and reaction-diffusion problems (Atangana, 2016). For advection-diffusion, CF solutions have been derived for kinetic adsorption in a one-dimensional column (Vieru et al., 2021), for two dimensions with time-dependent velocity and an instantaneous source (Kunasegaran and Mat Isa, 2026), and, with a Péclet-number analysis, in cylindrical geometry (Rubbab et al., 2021). These studies show that the CF operator can reproduce early arrival and heavy tails, but they also highlight that the memory parameter α must be calibrated against data.

It should be noted that the CF derivative has been the subject of some debate in the fractional calculus literature. Some authors have argued that the exponential kernel does not correspond to a true fractional derivative in the same sense as the power-law kernel,and that the CF operator can be expressed as a linear combination of integer-order integrals and derivatives (Giusti, 2018; Ortigueira and Tenreiro Machado, 2018). These criticisms are important and should not be dismissed. Nevertheless, the CF operator remains a useful tool for transport modeling because it provides a simple and analytically tractable way to incorporate fading memory into the governing equation. The exponential kernel can be interpreted as a memory that decays on a characteristic time scale, which is physically plausible for mass transfer between mobile and immobile zones. In this paper we use the CF derivative not as a replacement for the Caputo derivative in all contexts, but as a specific model for fading memory in one-dimensional contaminant transport.

Here we study the sim-infinite column with a constant-concentration inlet and examine its early response and mean breakthrough time. The contributions are: (i) an exact solution in the Laplace domain, validated against the Ogata-Banks formula for α=1; (ii) a closed-form expression for the early-time response and an exact result for the mean breakthrough time, which shows that α acts as a retardation factor; and (iii) a parametric study of the effect of α on breakthrough curves and concentration profiles. The paper is organized as follows. Section 2 presents the mathematical model, including the definition of CF derivative and the dimensionless form of the governing equation, and derives the exact solution in the Laplace domain. Section 3 presents numerical results and a parametric study. Section 4 concludes and outlines future work.

2. MATERIALS AND METHODS(MODEL DEVELOPMENT AND ANALYTICAL SOLUTION)

Consider a homogeneous, saturated porous column with uniform pore-water velocity v and longitudinal dispersion coefficient D. With a reference length L we introduce the dimensionless variables X=x/L, T=vt/L, c=C/C_0 and the Péclet number Pe=vL/D, where C_0 is the inlet concentration. Without sorption or decay, the classical ADE reads

$$\frac{\partial c}{\partial T} = \frac{1}{Pe}\frac{\partial^2 c}{\partial X^2} - \frac{\partial c}{\partial X} \quad (1)$$

The first term on the right-hand side represents longitudinal dispersion, and the second term represents advection. The Péclet number measures the relative importance of advection and dispersion: for large Pe, advection dominates and the front is sharp; for small Pe, dispersion dominates and the front is smooth. In the CF model the time derivative is replaced by the CF derivative of α (0< α<1):

Natural Resources for Human Health
eISSN: 2583-1194

$$ {}^{CF}D_T^{\alpha}\, c = \frac{1}{Pe}\frac{\partial^2 c}{\partial X^2} - \frac{\partial c}{\partial X} \tag{2} $$

with

$$ {}^{CF}D_T^{\alpha} f(T) = \frac{M(\alpha)}{1-\alpha}\int_0^T f\,'(\tau)\, e^{-\frac{\alpha(T-\tau)}{1-\alpha}} d\tau \tag{3} $$

where *M(a)* is a normalisation function, taken equal to 1 here(Caputo and Fabrizio, 2015; Losada and Nieto, 2015). The kernel $\exp[-\alpha(T-\tau)/(1-\alpha)]$ is exponential and non-singular. Its characteristic decay time is (1-α)/α in dimensionless units. For α close to 1, the decay time is small and the kernel approaches a Dirac delta, so the memory is short and the classical ADEis recovered. For α close to 0, the decay time is large and the memory extends far back in time, producing a pronounced tail in the breakthrough curve.

It is useful to compare the CF kernel with the power-law kernel of the Caputo derivative. TheCaputo kernel is $(T-\tau)^{-\alpha}/\Gamma(1-\alpha)$, which is singular at τ=T and has a heavy tail. The CFkernel is exponential and has a light tail. This difference has two consequences. First, the CF derivative produces a finite early-time response, whereas the Caputo derivative produces a response that is singular at T=0 for certain boundary conditions. Second, the CF derivative leads to a mean breakthrough time that is finite and exactly proportional to 1/α, whereas the Caputo derivative can produce a divergent mean for certain ranges of α. These properties make the CF model particularly convenient for analytical work. The memory in (2) acts on the concentration rate. An alternative is to place the memory in the diffusive flux, as in (Vieru et al., 2021), which gives a different model with differentphysical interpretation. Here we follow the more common approach of replacing the timederivative, because it preserves the structure of the ADE and allows a direct comparison with the classical solution. The initial and boundary conditions are

$$ c(X,0) = 0, \qquad c(0,T) = 1, \qquad c(\infty,T) = 0 \tag{4} $$

that is, a column initially free of contaminant, a constantconcentration at the inlet, and avanishing concentration far downstream. These conditions are the same as those used byOgata and Banks (1961), which makes the comparison between the classical and fractional solutions straightforward. It is worth noting that the CF derivative in (3) is written with f ' (τ) inside the integral. This form is equivalent to the original definition of Caputo and Fabrizio when f(0)=0, which is the casehere because c(X,0)=0. For nonzero initial conditions, the two forms differ by a term involvingf(0), and the Laplace transform must be modified accordingly. Since the initial condition in (4) is zero, the form (3) is appropriate and leads to the simple Laplace transformgiven below. The dimensionless memory time (1-α)/α can be interpreted as the time over which theconcentration rate retains memory of its past. For α=0.5, the memory time is 1, which iscomparable to the advective time scale L/v. For α=0.9, the memory time is approximately0.11, which is short compared with the advective time. This interpretation will be useful inSection 3 when we discuss the effect of α on breakthrough curves. With *M(a)* the Laplace transform of (3) is (Losada and Nieto, 2015)

$$ \mathcal{L}\{\ {}^{CF}D_T^{\alpha} f\} = \frac{s\tilde{f}(s) - f(0)}{(1-\alpha)s + \alpha} \tag{5} $$

Applying the Laplace transform in T to (2) and using c(X,0) = 0 gives the ordinary differential equation

$$ \frac{1}{\mathrm{Pe}}\frac{d^2\tilde{c}}{dX^2} - \frac{d\tilde{c}}{dX} - \Phi(s)\tilde{c} = 0 \tag{6} $$

with

$$ \Phi(s) = \frac{s}{(1-\alpha)s + \alpha} \tag{7} $$

Natural Resources for Human Health
eISSN: 2583-1194

Imposing $\tilde{c}(0,s) = 1/s$ and boundedness as $X \to \infty$ yields

$$\tilde{c}(X,s) = \frac{1}{s}\exp\left\{\frac{\mathrm{Pe}\,X}{2}\left[1-\sqrt{1+\frac{4\Phi(s)}{\mathrm{Pe}}}\right]\right\} \tag{8}$$

For α=1 we have $\Phi(s) = s$, and the inverse transform of (8) is the classical Ogata–Banks solution (Ogata and Banks, 1961)

$$c(X,T) = \frac{1}{2}[\mathrm{erfc}(\eta_-) + e^{\mathrm{Pe}\,X}\mathrm{erfc}(\eta_+)] \tag{9}$$

where $\eta_{\mp} = (X \mp T)/\left(2\sqrt{T/\mathrm{Pe}}\right)$. For 0<α<1 the inverse is computed numerically with theTalbot method (Talbot, 1979) in 30-digit arithmetic. For α=1 the numerical inversion reproduces (9) to machine precision (maximum absolute difference of 2 × 10^{-16} for 0.25 ≤ X ≤ 3 and 0.2 ≤ T ≤ 8). Two properties follow directly from (7) and (8). First, for$s \to \infty$ we have $\Phi(s) \to 1/(1-\alpha)$,and the initial value theorem gives the response immediately after the inlet is opened

$$c(X,0^+) = exp\left\{\frac{Pe\,X}{2}\left[1-\sqrt{1+\frac{4}{(1-\alpha)Pe}}\right]\right\} \tag{10}$$

This value is positive for every α <1 and vanishes only in the classical limit. It is a feature of the exponential kernel and can be read as a fast pathway for a fraction of the solute; its physical meaning has to be checked against tracer data.Second, writing (8) as $\tilde{c} = e^{g(s)}/s$, the mean breakthrough time at position X is

$$\langle T\rangle = \int_0^\infty [1-c(X,T)]\,dT = \frac{X}{\alpha} \tag{11}$$

because $g(0) = 0$ and $g'(0) = -X\Phi'(0) = -X/\alpha$. The order α therefore acts as a retardationfactor R=1/α: the mean arrival time is the advective time X multiplied by 1/α.

3. RESULTS AND DISCUSSION

All results use Pe=10 (for example v=0.5 m/day,L=100 m and D=5 m²/day, illustrative values) and the observation point X=1. Fig. 1 shows the breakthrough curves c(1,T). The markers, computed from the Ogata–Banks formula (9), coincide with the numerical curvefor α=1. Lowering α has three visible effects: a non-zero early response given by (10) (for α=0.5 the concentration at X=1 starts at 0.18); a slower approach to the inlet concentration; and a crossing of all the curves near $T \approx 0.8$.

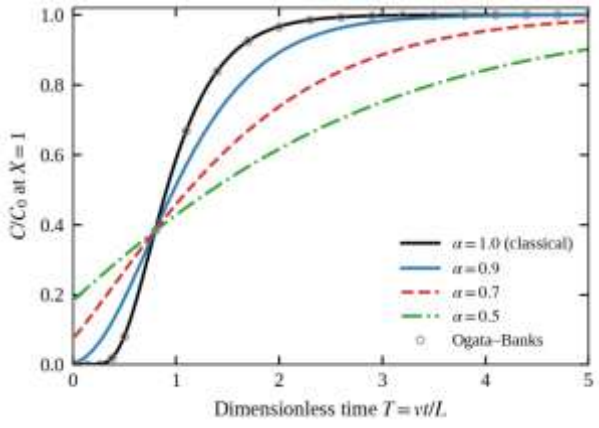


Fig. 1. Breakthrough curves at X=1 for Pe=10 and several orders α. Markers: Ogata–Banks solution

Table 1summarises the effect of α. The time T_{50}needed to reach c=0.5 grows from 0.91for α=1 to 1.35 for α=0.5, an increase of 48%, while the mean breakthrough time follows 1/ α exactly, as predicted by (11). The median grows more slowly than the mean, whichreflects the heavier tail of the CF breakthrough curves.

Table 1. Early response, half-arrival time and mean arrival time at X=1 (Pe=10)

α	$c(1,0^+)$	T_{50}	$\langle T \rangle = 1/\alpha$
1.0	0.000	0.91	1.00
0.9	0.002	0.98	1.11
0.8	0.026	1.05	1.25
0.7	0.072	1.12	1.43
0.6	0.126	1.22	1.67
0.5	0.181	1.35	2.00

Fig. 2 shows concentration profiles at T=1. The CF profiles are smoother: the concentration is lower near the classical front (c(1,1)=0.43 for α= 0.5 against 0.59 for α=1) but much higher farther downstream(c(2,1)=0.15 against 0.017, about nine times larger). At this time the contaminant has therefore travelled much farther than the classical ADE predicts.

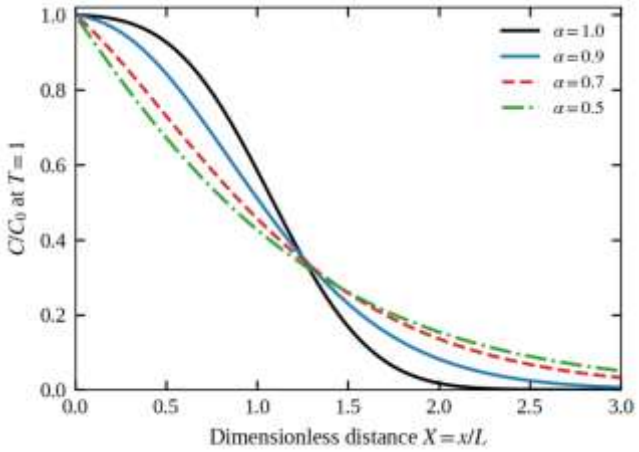


Fig. 2. Concentration profiles at T=1 for Pe=10 and several orders α.

4. CONCLUSIONS

A CF advection–dispersion model was solved exactly in the Laplace domain for a semi-infinite column. It reduces to the Ogata–Banks solution at α=1. For α < 1 the early response is nonzero and given in closed form, the mean breakthrough time is exactly 1/ α times the advective time, and the tail is heavier. The next steps involve sorption and first-order decay, fitting α to tracer-test data from Saharan aquifers, and a two-dimensional extension.

ETHICS STATEMENT: This research does not involve human or animals.

CONFLIT OF INTEREST: The authors have no conflit of interest.

FUNDING: This work received no specific funding.

REFERENCES

[1] Atangana, A., 2016. *On the new fractional derivative and application to nonlinear Fisher's reaction–diffusion equation*. Applied Mathematics and Computation. 273, 948–956. https://doi.org/10.1016/j.amc.2015.10.021

[2] Atangana, A., Baleanu, D., 2016. *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*. Thermal Science. 20, 763–769.

[3] Bear, J., 1972. Dynamics of Fluids in Porous Media. American Elsevier Publishing Company.

[4] Benson, D.A., Wheatcraft, S.W., Meerschaert, M.M., 2000. Application of a fractional advection-dispersion equation. Water Resources Research. 36, 1403–1412. https://doi.org/10.1029/2000WR900031

[5] Caputo, M., Fabrizio, M., 2015. *A new definition of fractional derivative without singular kernel*. Progress in Fractional Differentiation and Applications. 1, 73–85.

[6] De Moraes, R.M., Ozelim, L.C.D.S.M., Cavalcante, A.L.B., 2022. *Generalized Skewed Model for Spatial-Fractional Advective–Dispersive Phenomena*. Sustainability 14, 4024. https://doi.org/10.3390/su14074024

[7] Giusti, A., 2018. *A comment on some new definitions of fractional derivative*. Nonlinear Dynamics. 93, 1757–1763.

[8] Kunasegaran, P., Mat Isa, Z., 2026. *Analytical Solution of Two-Dimensional Advection-Diffusion Equation with Caputo-Fabrizio Derivative and Time Dependent Velocity*. Malaysian Journal of Fundamental and Applied Sciences. 22, 298–308. https://doi.org/10.11113/mjfas.v22n2.4531

[9] Losada, J., Nieto, J.J., 2015. Properties of a new fractional derivative without singular kernel. Progress in Fractional Differentiation and Applications. 1, 87–92.

[10] Ogata, A., and Banks, R., 1961, A solution of the differential equation of longitudinal dispersion in porous media: U.S. Geological Survey Professional Paper 411, p. A1-A7, https://doi.org/10.3133/pp411A

[11] Ortigueira, M.D., Tenreiro Machado, J., 2018. *A critical analysis of the Caputo–Fabrizio operator*. Communications in Nonlinear Science and Numerical Simulation. 59, 608–611. https://doi.org/10.1016/j.cnsns.2017.12.001

[12] Rubbab, Q., Nazeer, M., Ahmad, F., Chu, Y.-M., Ijaz Khan, M., Kadry, S., 2021. *Numerical simulation of advection–diffusion equation with caputo-fabrizio time fractional derivative in cylindrical domains: Applications of pseudo-spectral collocation method*. Alexandria Engineering Journal. 60, 1731–1738. https://doi.org/10.1016/j.aej.2020.11.022

[13] Schumer, R., Benson, D.A., Meerschaert, M.M., Baeumer, B., 2003. *Fractal mobile/immobile solute transport.*Water Resources Research. 39(10), 1296. https://doi.org/10.1029/2003WR002141

[14] Talbot, A., 1979. *The Accurate Numerical Inversion of Laplace Transforms.*IMA Journal of Applied Mathematics.23, 97–120. https://doi.org/10.1093/imamat/23.1.97

[15] Vieru, D., Fetecau, C., Ahmed, N., Shah, N.A., 2021. *A generalized kinetic model of the advection-dispersion process in a sorbing medium.*Mathematical Modelling of Natural Phenomena. 16, 39. https://doi.org/10.1051/mmnp/2021022